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SOME CONCEPTS OF MODERN ALGEBRAIC GEOMETRY: POINT, IDEAL AND HOMOMORPHISM

Martin Schlichenmaier

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Preface

This is a write-up of lectures given at the "Kleine Herbstschule 93" of the Graduiertenkolleg "Mathematik im Bereich Ihrer Wechselwirkungen mit der Physik" at the Ludwig-Maximilians-Universität München. Starting from classical algebraic geometry over the complex numbers (as it can be found for example in [GH]) it was the goal of these lectures to introduce some concepts of the modern point of view in algebraic geometry. Of course, it was quite impossible even to give an introduction to the whole subject in such a limited time. For this reason the lectures and now the write-up concentrate on the substitution of the concept of classical points by the notion of ideals and homomorphisms of algebras.

These concepts were established by Grothendieck in the 60s. In the following they were proven to be very fruitful in mathematics. I do not want to give an historic account of this claim. Let me just mention the proof of the Weil conjectures by Pierre Deligne (see [H,App.C]) and the three more recent results: Faltings' proof of Mordell's conjecture, Faltings' proof of the Verlinde formula and Wiles' work in direction towards Fermat's Last Theorem.[1] But also in theoretical physics, especially in connection with the theory of quantum groups and noncommutative geometries, it was necessary to extend the concept of points. This is one reason for the increasing interest in modern algebraic geometry among theoretical physicists. Unfortunately, to enter the field is

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[1]At the time this is written it is not clear whether the gap found in Wiles' "proof" really can be closed.

not an easy task. It has its own very well developed language and tools. To enter it in a linear way if it would be possible at all (which I doubt very much) would take a prohibitive long time. The aim of the lectures was to decrease the barriers at least a little bit and to make some appetite for further studies on a beautiful subject. I am aiming at mathematicians and theoretical physicists who want to gain some feeling and some understanding of these concepts. There is nothing new for algebraic geometers here.

What are the prerequisites? I only assume some general basics of mathematics (manifolds, complex variables, some algebra). I try to stay elementary and hence assume only few facts from algebraic geometry. All of these can be found in the first few chapters of [Sch].

The write-up follows very closely the material presented at the lectures. I did withstand the temptation to reorganize the material to make it more systematic, to supply all proofs, and to add other important topics. Especially the infinitesimal and the global aspects are still missing. Such an extension would considerably increase the amount of pages and hence obscure the initial goal to give a short introduction to the subject and to make appetite for further self-study. What made it easier for me to decide in this way is that there is a recent little book by Eisenbud and Harris available now [EH] which (at least that is what I hope) one should be able to study with profit after these lectures. The book [EH] substitutes (at least partially) the for a long period only available pedagogical introduction to the language of schemes, the famous red book of varieties and schemes by Mumford [Mu-1].[2] If you are looking for more details you can either consult Hartshorne [H] or directly Grothendieck [EGA I],[EGA]. Of course, other good sources are available now.

Finally, let me thank the audience for their active listening and the organizers of the Herbstschule for the invitation. It is a pleasure for me to give special thanks to Prof. M. Schottenloher and Prof. J. Wess.

1. Varieties

As we know from school the geometry of the plane consists of points, lines, curves, etc. with certain relations between them. The introduction of coordinates (i.e. numbers) to "name" the points has been proven to be very useful. In the real plane every point can be uniquely described by its pair (α, β) of Cartesian coordinates. Here α and β are real numbers. Curves are "certain" subset of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. The notion "certain" is of

[2] Which is still very much recommended to be read. Recently, it has been reprinted in the Springer Lecture Notes Series.

course very unsatisfactory.

In classical algebraic geometry the subsets defining the geometry are the set of points where a given set of polynomials have a common zero (if we plug in the coordinates of the points in the polynomial). To give an example: the polynomials X and Y are elements of the polynomial ring in 2 variables over the real numbers $\mathbb{R}$. They define the following polynomial functions:

$$X,\ Y:\ \mathbb{R}^2 \to \mathbb{R}, \qquad (\alpha,\beta) \mapsto X(\alpha,\beta) = \alpha, \quad \text{resp.} \quad Y(\alpha,\beta) = \beta\ .$$

These two functions are called coordinate functions. The point $(\alpha_0, \beta_0) \in \mathbb{R}^2$ can be given as zero set

$$\{(\alpha,\beta) \in \mathbb{R}^2 \mid X(\alpha,\beta) - \alpha_0 = 0,\ Y(\alpha,\beta) - \beta_0 = 0\ \}\ .$$

Let me come to the general definition. For this let $\mathbb{K}$ be an arbitrary field (e.g. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_p$, $\overline{\mathbb{F}_p}, \dots$) and $\mathbb{K}^n = \underbrace{\mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}}_{n \text{ times}}$ the $n-$dimensional *affine space* over $\mathbb{K}$. I shall describe the objects of the geometry as zero sets of polynomials. For this let $R_n = \mathbb{K}[X_1, X_2, \dots, X_n]$ be the polynomial ring in n variables. A subset A of $\mathbb{K}^n$ should be a geometric object if there exist finitely many polynomials $f_1, f_2, \dots, f_s \in R_n$ such that

$$\mathbf{x} \in A \qquad \text{if and only if} \qquad f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_s(\mathbf{x}) = 0\ .$$

Here and in the following it is understood that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{K}^n$ and $f(\mathbf{x}) \in \mathbb{K}$ denotes the number obtained by replacing the variable X_1 by the number x_1, etc..

Using the notion of ideals it is possible to define these sets A in a more elegant fashion. An ideal of an arbitrary ring R is a subset of R which is closed under addition : $I + I \subseteq I$, and under multiplication with the whole ring: $R \cdot I \subseteq I$. A good reference to recall the necessary prerequisites from algebra is [Ku]. Now let $I = (f_1, f_2, \dots, f_r)$ be the ideal generated by the polynomials $f_1, f_2, \dots, f_s$ which define A, e.g.

$$I = R \cdot f_1 + R \cdot f_2 + \cdots + R \cdot f_s = \{r_1 f_1 + r_2 f_2 + \cdots + r_s f_s \mid r_i \in R,\ i = 1, \dots, s\}\ .$$

Definition. A subset A of $\mathbb{K}^n$ is called an *algebraic set* if there is an ideal I of R_n such that

$$\mathbf{x} \in A \iff f(\mathbf{x}) = 0 \quad \text{for all } f \in I.$$

The set A is called the *vanishing set of the ideal* I, in symbols $A = V(I)$ with

$$V(I) := \{\, \mathbf{x} \in \mathbb{K}^n \mid f(\mathbf{x}) = 0,\ \forall f \in I\ \}\ . \tag{1-1}$$

Remark 1. It is enough to test the vanishing with respect to the generators of the ideal in the definition.

Remark 2. There is no finiteness condition mentioned in the definition. Indeed this is not necessary, because the polynomial ring R_n is a noetherian ring. Recall a ring is a *noetherian ring* if every ideal has a finite set of generators. There are other useful equivalent definitions of a noetherian ring. Let me here recall only the fact that every strictly ascending chain of ideals (starting from one ideal) consists only of finitely many ideals. But every field $\mathbb{K}$ has only the (trivial) ideals $\{0\}$ and $\mathbb{K}$ (why?), hence $\mathbb{K}$ is noetherian. Trivially, all principal ideal rings (i.e. rings where every ideal can be generated by just one element) are noetherian. Beside the fields there are two important examples of principal ideal rings: $\mathbb{Z}$ the integers, and $\mathbb{K}[X]$ the polynomial ring in one variable over the field $\mathbb{K}$. Let me recall the proof for $\mathbb{Z}$. Take I an ideal of $\mathbb{Z}$. If $I = \{0\}$ we are done. Hence assume $I \neq \{0\}$ then there is a $n \in \mathbb{N}$ with $n \in I$ minimal. We now claim $I = (n)$. To see this take $m \in I$. By the division algorithm of Euklid there are $q, r \in \mathbb{Z}$ with $0 \leq r < n$ such that $m = qn + r$. Hence, with m and n in I we get $r = m - qn \in I$. But n was chosen minimal, hence $r = 0$ and $m \in (n)$. Note that the proof for $\mathbb{K}[X]$ is completely analogous if we replace the division algorithm for the integers by the division algorithm for polynomials.

Now we have

Hilbertscher Basissatz. *Let R be a noetherian Ring. Then $R[X]$ is also noetherian.*

As a nice exercise you may try to proof it by yourself (maybe guided by [Ku]).

Remark 3. If R is a noncommutative ring one has to deal with left, right and two-sided ideals. It is also necessary to define left, right, and two-sided noetherian.

It is time to give some **examples of algebraic sets**:

(**1**) The whole affine space is the zero set of the zero ideal: $\mathbb{K}^n = V(0)$.

(**2**) The empty set is the zero set of the whole ring R_n: $\emptyset = V((1))$.

(**3**) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{K}^n$ be a point given by its coordinates. Define the ideal

$$I_\alpha = (X_1 - \alpha_1, X_2 - \alpha_2, \ldots, X_n - \alpha_n),$$

then $\{\alpha\} = V(I_\alpha)$.

(**4**) Now take 2 points α, β and their associated ideals I_α, I_β as defined in (3). Then $I_\alpha \cap I_\beta$ is again an ideal and we get $\{\alpha, \beta\} = V(I_\alpha \cap I_\beta)$.
This is a **general fact**. Let $A = V(I)$ and $B = V(J)$ be two algebraic sets then the union $A \cup B$ is again an algebraic set because $A \cup B = V(I \cap J)$. Let me give a proof of this. Obviously, we get for two ideals K and L with $K \subseteq L$ for their vanishing sets

$V(K) \supseteq V(L)$. Hence because $I \cap J \subseteq I$ and $I \cap J \subseteq J$ we obtain $V(I \cap J) \supseteq V(I) \cup V(J)$. To proof the other inclusion assume that there is an $x \notin V(I) \cup V(J)$ then there are $f \in I$ and $g \in J$ with $f(x) \neq 0$ and $g(x) \neq 0$. Now $f \cdot g \in I \cap J$ but $(f \cdot g)(x) = f(x) \cdot g(x) \neq 0$. Hence $x \notin V(I \cap J)$. Let me repeat the result for further reference:

$$V(I) \cup V(J) \quad = \quad V(I \cap J)\ . \tag{1-2}$$

(**5**) A hypersurface H is the vanishing set of the ideal generated by a single polynomial f: $H = V((f))$. An example in $\mathbb{C}^2$ is given by $I = (Y^2 - 4X^3 + g_2X + g_3)$ where $g_2, g_3 \in \mathbb{C}$. The set $V(I)$ defines a cubic curve in the plane. For general g_2, g_3 this curve is isomorphic to a (complex) one-dimensional torus with the point 0 removed.

(**6**) Linear affine subspaces are algebraic sets. A linear affine subspace of $\mathbb{K}^n$ is the set of solutions of a system of linear equations $A \cdot \mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} \mathbf{a}_{1,*} \\ \cdots \\ \mathbf{a}_{r,*} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \cdots \\ b_r \end{pmatrix}, \quad \mathbf{a}_{i,*} \in \mathbb{K}^n, \quad b_i \in \mathbb{K}\ , i = 1, \dots, r\ .$$

The solutions (by definition) are given as the elements of the vanishing set of the ideal

$$I = (\mathbf{a}_{1,*} \cdot X - b_1,\ \mathbf{a}_{2,*} \cdot X - b_2,\ \cdots,\ \mathbf{a}_{r,*} \cdot X - b_r)\ .$$

(**7**) A special case are the straight lines in the plane. For this let $l_i = a_{i,1}X + a_{i,2}Y - b_i$, $i = 1, 2$ be two linear forms. Then $L_i = V((l_i))$, $i = 1, 2$ are lines. For the union of the two lines we obtain by (1-2)

$$L_1 \cup L_2 = V((l_1) \cap (l_2)) = V((l_1 \cdot l_2))\ .$$

Note that I do not claim $(l_1) \cap (l_2) = (l_1 \cdot l_2)$. The reader is encouraged to search for conditions when this will hold. For the intersection of the two lines we get $L_1 \cap L_2 = V((l_1, l_2))$ which can be written as $V((l_1) + (l_2))$. Of course, this set consists just of one point if the linear forms l_1 and l_2 are linearly independent. Again, there is the **general fact**

$$V(I) \cap V(J) = V(I + J), \tag{1-3}$$

where

$$I + J := \{\, f + g \mid f \in I, g \in J \,\}\ .$$

You see there is a ample supply of examples for algebraic sets. Now we introduce for $\mathbb{K}^n$ a topology, the *Zariski-Topology*. For this we call a subset U open if it is a complement of an algebraic set, i.e. $U = \mathbb{K}^n \setminus V(I)$ where I is an ideal of R_n. In other words: the closed sets are the algebraic sets. It is easy to verify the axioms for a topology:

(1) $\mathbb{K}^n$ and $\emptyset$ are open.
(2) Finite intersections are open:

$$U_1 \cap U_2 = (\mathbb{K}^n \setminus V(I_1)) \cap (\mathbb{K}^n \setminus V(I_2)) = \mathbb{K}^n \setminus (V(I_1) \cup V(I_2)) = \mathbb{K}^n \setminus V(I_1 \cap I_2) \ .$$

(3) Arbitrary unions are open:

$$\bigcup_{i \in S} (\mathbb{K}^n \setminus V(I_i)) = \mathbb{K}^n \setminus \bigcap_{i \in S} V(I_i) = \mathbb{K}^n \setminus V(\sum_{i \in S} I_i) \ .$$

Here S is allowed to be an infinite index set. The ideal $\sum_{i \in S} I_i$ consists of elements in R_n which are finite sums of elements belonging to different I_i. The claim (1-3) easily extends to this setting.

Let us study the affine line $\mathbb{K}$. Here $R_1 = \mathbb{K}[X]$. All ideals in $K[X]$ are principal ideals, i.e. generated by just one polynomial. The vanishing set of an ideal consists just of the finitely many zeros of this polynomial (if it is not identically zero). Conversely, for every set of finitely many points there is a polynomial vanishing exactly at these points. Hence, beside the empty-set and the whole line the algebraic sets are the sets of finitely many points. At this level there is already a new concept showing up. The polynomial assigned to a certain point set is not unique. For example it is possible to increase the vanishing order of the polynomial at a certain zero without changing the vanishing set. It would be better to talk about point sets with multiplicities to get a closer correspondence to the polynomials. Additionally, if $\mathbb{K}$ is not algebraically closed then there are non-trivial polynomials without any zero at all. These ideas we will take up in later lectures. The other important observation is that the open sets in $\mathbb{K}$ are either empty or dense. The latter says that the closure $\overline{U}$ of U, i.e. the smallest closed set which contains U is the whole space $\mathbb{K}$. Assuming the whole space to be irreducible this is true in a more general context.

Definition.
(a) Let V be a closed set. V is called *irreducible* if for every decomposition $V = V_1 \cup V_2$ with V_1, V_2 closed we have $V_1 = V$ or $V_2 = V$.
(b) An algebraic set which is irreducible is called a *variety*.

Now let U be an open subset of an irreducible V. The two set $V \setminus U$ and $\overline{U}$ are closed and $V = (V \setminus U) \cup \overline{U}$. Hence, V has to be one of these sets. Hence, either $U = \emptyset$ or $V = \overline{U}$. As promised, this shows that every open subset of an irreducible space is either empty or dense. Note that this has nothing to do with our special situation. It follows from general topological arguments. In the next section we will see that the spaces $\mathbb{K}^n$ are irreducible.

Up to now we were able to describe our geometric objects with the help of the ring of polynomials. This ring plays another important role in the whole theory. We need it to study polynomial (algebraic) functions on $\mathbb{K}^n$. If $f \in R_n$ is a polynomial then $\mathbf{x} \mapsto f(\mathbf{x})$ defines a map from $\mathbb{K}^n$ to $\mathbb{K}$. This can be extended to functions on algebraic sets $A = V(I)$. We associate to A the quotient ring

$$R(A) := \mathbb{K}[X_1, X_2, \ldots, X_n]/I \ .$$

This ring is called the coordinate ring of A. The elements of $R(A)$ can be considered as functions on A. Take $\mathbf{x} \in A$, and $\bar{f} \in R(A)$ then $\bar{f}(\mathbf{x}) := f(\mathbf{x})$ is a well-defined element of $\mathbb{K}$. Assume $\bar{f} = \bar{g}$ then there is an $h \in I$ with $f = g + h$ hence $f(\mathbf{x}) = g(\mathbf{x}) + h(\mathbf{x}) = g(\mathbf{x}) + 0$. You might have noticed that it is not really correct to call this ring the coordinate ring of A. It is not clear, in fact it is not even true that the ideal I is fixed by the set A. But $R(A)$ depends on I. A first way to avoid these complications is to assign to every A a unique defining ideal,

$$I(A) := \{f \in R_n \mid f(\mathbf{x}) = 0, \forall \mathbf{x} \in A\} \ . \tag{1-4}$$

It is the largest ideal which defines A. For arbitrary ideals we always obtain $I(V(I)) \supseteq I$.

There is a second possibility which even takes advantage out of the non-uniqueness. We could have added the additional data of the defining ideal I in the notation. Just simply assume that when we use A it comes with a certain I. Compare this with the situation above where we determined the closed sets of $\mathbb{K}$. Again this at the first glance annoying fact of non-uniqueness of I will allow us to introduce multiplicities in the following which in turn will be rather useful as we will see.

Here another warning is in order. The elements of $R(A)$ define usual functions on the set A. But different elements can define the same function. In particular, $R(A)$ can have zero divisors and nilpotent elements (which always give the zero function).

The ring $R(A)$ contains all the geometry of A. As an example, take A to be a curve in the plane and P a point in the plane. Then $A = V((f))$ with f a polynomial in X and Y and $P = V((X - \alpha, Y - \beta))$. Now $P \subset A$ (which says that the point P lies on A) if and only if $(X - \alpha, Y - \beta) \supset (f)$. Moreover, in this case we obtain the following diagram of ring homomorphisms

$$\begin{array}{ccc}
R_2 & \xrightarrow{/(f)} & R(A) \\
\subseteq\uparrow & & \subseteq\uparrow \\
(X-\alpha, Y-\beta) & \xrightarrow{/(f)} & (X-\alpha, Y-\beta)/(f) \\
\subseteq\uparrow & & \subseteq\uparrow \\
(f) & \longrightarrow & \{0\}\ .
\end{array}$$

The quotient $(X-\alpha, Y-\beta)/(f)$ is an ideal of $R(A)$ and corresponds to the point P lying on A.

Indeed, this is the general situation which we will study in the following sections: the algebraic sets on A correspond to the ideals of $R(A)$ which in turn correspond to the ideals lying between the defining ideal of A and the whole ring R_n.

Let me close this section by studying the geometry of a single point $P = (\alpha, \beta) \in \mathbb{K}^2$. A defining ideal is $I = (X-\alpha, Y-\beta)$. If we require "multiplicity one" this is the defining ideal. Hence, the coordinate ring $R(P)$ of a point is $\mathbb{K}[X,Y]/I \cong \mathbb{K}$. The isomorphismus is induced by the homomorphism $\mathbb{K}[X,Y] \to \mathbb{K}$ given by $X \to \alpha,\ Y \to \beta$. Indeed, every element r of $\mathbb{K}[X,Y]$ can be given as

$$r = r_0 + (X-\alpha)\cdot g + (Y-\beta)\cdot f, \quad r_0 \in \mathbb{K},\ f,g \in \mathbb{K}[X,Y]\ . \tag{1-5}$$

Under the homomorphism r maps to r_0 . Hence r is in the kernel of the map if and only if r_0 equals 0 which in turn is the case if and only if r is in the ideal I. The description (1-5) also shows that I is a maximal ideal. We call an ideal I a maximal ideal if there are no ideals between I and the whole ring R (and $I \neq R$). Any ideal strictly larger than the above I would contain an r with $r_0 \neq 0$. Now this ideal would contain $r, (X-\alpha), (Y-\beta)$ hence also r_0. Hence also $(r_0)^{-1}\cdot r_0 = 1$. But an ideal containing 1 is always the whole ring.

On the geometric side the points are the minimal sets. On the level of the ideals in R_n this corresponds to the fact that an ideal defining a point (with multiplicity one) is a maximal ideal. If the field $\mathbb{K}$ is algebraically closed then every maximal ideal corresponds indeed to a point.

2. The spectrum of a ring

In the last lecture we saw that geometric objects are in correspondence to algebraic objects of the coordinate ring. This we will develop more systematically in this lecture. We had the following correspondences (1-1), (1-4)

$$\text{ideals of } R_n \quad \xrightarrow{V} \quad \text{algebraic sets}$$

$$\text{ideals of } R_n \quad \xleftarrow{I} \quad \text{algebraic sets.}$$

Recall the definitions: ($R_n = \mathbb{K}[X_1, X_2, \dots, X_n]$)

$$V(I) := \{\, \mathbf{x} \in \mathbb{K}^n \mid f(\mathbf{x}) = 0,\ \forall f \in I \,\}, \qquad I(A) := \{\, f \in R_n \mid f(\mathbf{x}) = 0, \forall \mathbf{x} \in A \,\}\ .$$

In general $I(V(I))$ will be bigger than the ideal I. Let me give an example. Consider in $\mathbb{C}[X]$ the ideals $I_1 = (X)$ and $I_2 = (X^2)$. Then $V(I_1) = V(I_2) = \{0\}$. Hence both ideals define the same point as vanishing set. Moreover $I(V(I_2)) = I_1$ because I_1 is a maximal ideal. If we write down the coordinate ring of the two situations we obtain for I_1 the ring $\mathbb{C}[X]/(X) \cong \mathbb{C}$. This is the expected situation because the functions on a point are just the constants. For I_2 we obtain $\mathbb{C}[X]/(X^2) \cong \mathbb{C} \oplus \mathbb{C} \cdot \epsilon$ the algebra generated by 1 and ϵ with the relation $\epsilon^2 = 0$ (X maps to ϵ). Hence, there is no 1-1 correspondence between ideals and algebraic sets. If one wants such a correspondence one has to throw away the "wrong" ideals. This is in fact possible (by considering the so called radical ideals, see the definition below). Indeed, it is rather useful to allow all ideals to obtain more general objects (which are very useful) than the classical objects.

To give an example: take the affine real line and let $I_t = (X^2 - t^2)$ for $t \in \mathbb{R}$ be a family of ideals. The role of t is the role of a parameter one is allowed to vary. Obviously,

$$I_t = ((X - t)(X + t)) = (X - t) \cdot (X + t).$$

For $t \neq 0$ we obtain $V(I_t) = \{t, -t\}$ and for $t = 0$ we obtain $V(I_0) = \{0\}$. We see that for general values of t we get two points, and for the value $t = 0$ one point. If we approach with t the value 0 the two different points $\pm t$ come closer and closer together. Now our intuition says that the limit point $t = 0$ better should be counted twice. This intuition we can make mathematically precise on the level of the coordinate rings. Here we have

$$R_t = \mathbb{R}[X]/I_t \cong \mathbb{R} \oplus \mathbb{R} \cdot \epsilon, \quad \epsilon^2 = t^2\ .$$

The coordinate ring is a two-dimensional vector space over $\mathbb{R}$ which reflects the fact that we deal with two points. Everything here is also true for the exceptional value $t = 0$. Especially R_0 is again two-dimensional. This says we count the point $\{0\}$ twice.

The drawback is that the interpretation of the elements of R_t as classical functions will not be possible in all cases. In our example for $t = 0$ the element $\bar{X}$ will be nonzero but $\bar{X}(0) = 0$.

For the following definitions let R be an arbitrary commutative ring with unit 1.

Definition.

(a) An ideal P of R is called a *prime ideal* if $P \neq R$ and $a \cdot b \in P$ implies $a \in P$ or $b \in P$.

(b) An ideal M of R is called a *maximal ideal* if $M \neq R$ and for every ideal M' with $M' \supseteq M$ it follows that $M' = M$ or $M' = R$.

(c) Let I be an ideal. The *radical* of I is defined as

$$\operatorname{Rad}(I) := \{\, f \in R \mid \exists n \in \mathbb{N} : f^n \in I \,\} \ .$$

(d) The *nil radical* of the ring R is defined as $\operatorname{nil}(R) := \operatorname{Rad}(\{0\})$.

(e) A ring is called *reduced* if $\operatorname{nil}(R) = \{0\}$.

(f) An ideal I is called a *radical ideal* if $\operatorname{Rad}(I) = I$.

Starting from these definitions there are a lot of easy exercises for the reader:

(1) Let P be a prime ideal. Show: R/P is a ring without zero divisor (such rings are called integral domains).

(2) Let M be a maximal ideal. Show R/M is a field.

(3) Every maximal ideal is a prime ideal.

(4) $\operatorname{Rad}(I)$ is an ideal.

(5) $\operatorname{Rad}(I)$ equals the intersection of all prime ideals containing I.

(6) $\operatorname{nil}(R/I) = \operatorname{Rad}(I)/I$ and conclude that every prime ideal is a radical ideal.

(7) $\operatorname{Rad} I$ is a radical ideal.

Let me return to the rings R_t defined above. The ideals I_t are not prime because neither $X + t$ nor $X - t$ are in I_t but $(X + t)(X - t) \in I_t$. In particular, R_t is not an integral domain: $(\epsilon + t)(\epsilon - t) = 0$. Let us calculate $\operatorname{nil}(R_t)$. For this we take an element $0 \neq z = a + b\epsilon$ and calculate

$$0 = (a + b\epsilon)^n = a^n + \binom{n}{1} a^{n-1} b^1 \epsilon + \binom{n}{2} a^{n-2} b^2 \epsilon^2 + \cdots \ .$$

Replacing ϵ^2 by the positive real number t^2 we obtain

$$0 = (a+b\epsilon)^n = \left(a^n + \binom{n}{2} a^{n-2} b^2 t^2 + \binom{n}{4} a^{n-4} b^4 t^4 + \cdots \right) +$$
$$+\epsilon \left(\binom{n}{1} a^{n-1} b^1 + \binom{n}{3} a^{n-3} b^3 t^2 + \cdots \right) .$$

From this we conclude that all terms in the first and in the second sum have to vanish (all terms have the same sign). This implies $a = 0$. Regarding the last element in both sums we see that for $t \neq 0$ we get $b = 0$. Hence $\mathrm{nil}(R_t) = \{0\}$, for $t \neq 0$ and the ring R_t is reduced. For $t = 0$ the value of b is arbitrary. Hence $\mathrm{nil}(R_0) = (\epsilon)$, which says that R_0 is not a reduced ring. This is the typical situation: a non-reduced coordinate ring $R(V)$ corresponds to a variety V which should be considered with higher multiplicity.

For the polynomial ring we have the following very important result.

Hilbertscher Nullstellensatz. *Let I be an ideal in $R_n = \mathbb{K}[X_1, X_2, \ldots, X_n]$. If $\mathbb{K}$ is algebraically closed then $I(V(I)) = \mathrm{Rad}(I)$.*

The proof of this theorem is not easy. The main tool is the following version of the Nullstellensatz which more resembles his name

Hilbertscher Nullstellensatz. *Let I be an ideal in $R_n = \mathbb{K}[X_1, X_2, \ldots, X_n]$. $I \neq R_n$. If $\mathbb{K}$ is algebraically closed then $V(I) \neq \emptyset$. In other words given a set of polynomials such that the constant polynomial 1 cannot be represented as a R_n-linear sum in these polynomials then there is a simultaneous zero of these polynomials.*

For the proof let me refer to [Ku].

The Nullstellensatz gives us a correspondence between algebraic sets in $\mathbb{K}^n$ and the radical ideals of $R_n = \mathbb{K}[X_1, X_2, \ldots, X_n]$. If we consider the prime ideals we get

Proposition. *Let P be a radical ideal. Then P is a prime ideal if and only if $V(P)$ is a variety.*

Before we come to the proof of the proposition let me state the following simple observation. For arbitrary subsets S and T of $\mathbb{K}^n$ the ideals $I(S)$ and $I(T)$ can be defined completely in the same way as in (1-4), i.e.,

$$I(S) := \{f \in R_n \mid f(\mathbf{x}) = 0, \forall \mathbf{x} \in S\} . \tag{2-1}$$

It is easy to show that

$$I(S \cup T) = I(S) \cap I(T), \quad \text{and} \quad V(I(S)) = \overline{S} . \tag{2-2}$$

Here $\overline{S}$ denotes the topological closure of S, which is the smallest (Zariski-)closed subset of $\mathbb{K}^n$ containing S.

Proof of the above proposition. Let P be a prime ideal and set $Y = V(P)$ then $I(V(P)) = \mathrm{Rad}(P) = P$ by the Nullstellensatz. Assuming $Y = Y_1 \cup Y_2$ a closed decomposition of Y then $I(Y) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2) = P$. Because P is prime either $P = I(Y_1)$ or $P = I(Y_2)$. Assume the first then $Y_1 = V(I(Y_1)) = V(P) = Y$ (using that Y_1 is closed).
Conversely: let $Y = V(P)$ be irreducible with P a radical ideal. By the Nullstellensatz $P = \mathrm{Rad}(P) = I(Y)$. Let $f \cdot g \in P$ then $f \cdot g$ vanishes on Y. We can decompose $Y = (Y \cap V(f)) \cup (Y \cap V(g))$ into closed subset of Y. By the irreducibility it has to coincide with one of them. Assume with the first. But this implies that $V(f) \supseteq Y$ and hence f is identically zero on Y. We get $f \in I(Y) = P$. This shows that P is a prime ideal. $\square$

Note the fact that we restricted the situation to radical ideals corresponds to the fact that varieties as sets have always multiplicity 1, hence they are always “reduced”. To incorporate all ideals and hence “nonreduced structures” we have to use the language of schemes (see below).

Let us look at the maximal ideals of $R_n = \mathbb{K}[X_1, X_2, \ldots, X_n]$. (Still $\mathbb{K}$ is assumed to be algebraically closed). The same argument as in the two-dimensional case shows that the ideals

$$M_\alpha = (X_1 - \alpha_1,\ X_2 - \alpha_2,\ \ldots\ ,\ X_n - \alpha_n)$$

are maximal and that $R_n/M_\alpha \cong \mathbb{K}$. This is even true if the field $\mathbb{K}$ is not algebraically closed. Now let M' be a maximal ideal. By the Nullstellensatz (here algebraically closedness is important) there is a common zero α for all elements $f \in M'$. Take $f \notin M'$ then $R_n = (f, M')$. Now $f(\alpha) = 0$ would imply that α is a zero of all polynomials in R_n which is impossible. Hence, every polynomial f which vanishes at α lies in M'. All elements in M_α have α as a zero. This implies $M_\alpha \subseteq M' \subsetneqq R_n$. By the maximality of M_α we conclude $M_\alpha = M'$.

Everything can be generalized to an arbitrary variety A over an algebraically closed field. The points of A correspond to the maximal ideals of R_n lying above the defining prime ideal P of A. They correspond exactly to the maximal ideals in $R(A)$. All of them can be given as M_α/P. This can be extended to the varieties of $\mathbb{K}^n$ lying on A. They correspond to the prime ideals of R_n lying between the prime ideal P and the whole ring. They in turn can be identified with the prime ideals of $R(A)$.

Coming back to arbitrary rings it is now quite useful to talk about dimensions.

Definition. Let R be a ring. The *(Krull-) dimension* $\dim R$ of a ring R is defined as the maximal length r of all strict chains of prime ideals P_i in R

$$P_0 \subsetneqq P_1 \subsetneqq P_2 \ldots \subsetneqq P_r \subsetneqq R\ .$$

Example 1. For a field $\mathbb{K}$ the only (prime) ideals are $\{0\} \subset \mathbb{K}$. Hence $\dim \mathbb{K} = 0$.

Example 2. The dimension of $R_n = \mathbb{K}[X_1, X_2, \ldots, X_n] = R(\mathbb{K}^n)$ is n. This result one should expect from a reasonable definition of dimension. Indeed we have the chain of prime ideals

$$(0) \subsetneqq (X_1) \subsetneqq (X_1, X_2) \subsetneqq \cdots \subsetneqq (X_1 - \alpha_1,\ X_2 - \alpha_2,\ \ldots,\ X_n - \alpha_n) \subsetneqq R_n\ .$$

Hence $\dim R_n \geq n$. With some more commutative algebra it is possible to show the equality, see [Ku,S.54].

Example 3. As a special case one obtains $\dim \mathbb{K}[X] = 1$. Here the reason is a quite general result. Recall that $\mathbb{K}[X]$ is a principal ideal ring without zero divisors. Hence, every ideal I can be generated by one element f. Assume I to be a prime ideal, $I \neq \{0\}$ and let $M = (g)$ be a maximal ideal lying above I. We show that I is already maximal. Because $(f) \subseteq (g)$ we get $f = r \cdot g$. But I is prime. This implies either r or g lies in I. If $g \in I$ we are done. If $r \in I$ then $r = s \cdot f$ and $f = f \cdot s \cdot g$. In a ring without zero divisor one is allowed to cancel common factors. We obtain $1 = s \cdot g$. Hence, $1 \in M$ which contradicts the fact that M is not allowed to be the whole ring. From this it follows that $\dim \mathbb{K}[X] = 1$. Note that we did not make any reference to the special nature of the polynomial ring here.

What are the conditions on f assuring that the ideal (f) is prime. The necessary and sufficient condition is that f is irreducible but not a unit. This says if there is decomposition $f = g \cdot h$ then either g or h has to be a unit (i.e. to be invertible) which in our situation says that g or h must be a constant. This can be seen in the following way. From the decomposition it follows (using (f) is prime) that either g or h has to be in (f) hence is a multiple of f. By considering the degree we see that the complementary factor has degree zero and hence is a constant.
Conversely, let f be irreducible but not a unit. Assume $g \cdot h \in (f)$, then $g \cdot h = f \cdot r$. In the polynomial ring we have unique factorization (up to units) into irreducible elements. Hence, the factor f is contained either in g or h. This shows the claim.

Example 4. The ring of integers $\mathbb{Z}$ is also a principal ideal ring without zero divisor. Again we obtain $\dim \mathbb{Z} = 1$. In fact, the integers behave very much (at least from the point of view of algebraic geometry) like the affine line over a field. What are the "points" of $\mathbb{Z}$? As already said the points should correspond to the maximal ideals. Every prime ideal in $\mathbb{Z}$ is maximal. An ideal (n) is prime exactly if n is a prime number. Hence, the "points" of $\mathbb{Z}$ are the prime numbers.

Now we want to introduce the Zariski topology on the set of all prime ideals of a ring. First we introduce the sets

$$\begin{aligned} \mathrm{Spec}(R) &:= \{ P \mid P \text{ is a prime ideal of } R \}, \\ \mathrm{Max}(R) &:= \{ P \mid P \text{ is a maximal ideal of } R \} . \end{aligned}$$

The set $\mathrm{Spec}(R)$ contains in some sense all irreducible "subvarieties" of the "geometric model" of R. Let S be an arbitrary subset of R. We define the associated subset of $\mathrm{Spec}(R)$ as the set consisting of the prime ideals which contain S:

$$V(S) := \{ P \in \mathrm{Spec}(R) \mid P \supseteq S \} . \tag{2-3}$$

The subsets of $\mathrm{Spec}(R)$ obtained in this way are called the closed subsets. It is obvious that $S \subseteq T$ implies $V(S) \supseteq V(T)$. Clearly, $V(S)$ depends only of the ideal generated by S: $V((S)) = V(S)$.

This defines a topology on $\mathrm{Spec}(R)$ the *Zariski topology*.

(1) The whole space and the empty set are closed: $V(0) = \mathrm{Spec}(R)$ and $V(1) = \emptyset$.

(2) Arbitrary intersections of closed sets are again closed:

$$\bigcap_{i \in J} V(S_i) = V(\bigcup_{i \in J} S) . \tag{2-4}$$

(3) Finite unions of closed set are again closed:

$$V(S_1) \cup V(S_2) = V((S_1) \cap (S_2)) . \tag{2-5}$$

Let me just show (2-5) here. Because $(S_1), (S_2) \supseteq (S_1) \cap (S_2)$ we get $V(S_1) \cup V(S_2) \subseteq V((S_1) \cap (S_2))$. Take $P \in V((S_1) \cap (S_2))$.This says $P \supseteq (S_1) \cap (S_2)$. If $P \supseteq (S_1)$ we get $P \in V(S_1)$ and we are done. Hence, assume $P \not\supseteq (S_1)$. Then there is a $y \in (S_1)$ such that $y \notin P$. But now $y \cdot (S_2)$ is a subset of both (S_1) and (S_2) because they are ideals. Hence, $y \cdot (S_2) \subseteq P$. By the prime ideal condition $(S_2) \subseteq P$ which we had to show. □

Remark 1. The closed points in $\mathrm{Spec}(R)$ are the prime ideals which are maximal ideals.

Remark 2. If we take any prime ideal P then the (topological) closure of P in $\mathrm{Spec}(R)$ is given as

$$V(P) = \{ Q \in \mathrm{Spec}(R) \mid Q \supseteq P \} .$$

Hence, the closure of P consists of P and all "subvarieties" of P together. In particular the closure of a curve consists of the curve as geometric object and all points lying on the curve.

At the end of this lecture let me return to the affine line over a field $\mathbb{K}$, resp. its algebraic model the polynomial ring in one variable $\mathbb{K}[X]$. We saw already that we have the non-closed point corresponding to the prime ideal $\{0\}$ and the closed points corresponding to the prime ideals (f) (which are automatically maximal) where f is an irreducible polynomial of degree ≥ 1. If $\mathbb{K}$ is an algebraically closed field the only irreducible polynomials are the linear polynomials $X - \alpha$. Hence, the closed points of $\operatorname{Spec}(\mathbb{K}[X])$ indeed correspond to the geometric points $\alpha \in \mathbb{K}$. The non-closed point corresponds to the whole affine line.

Now we want to drop the condition that $\mathbb{K}$ is algebraically closed. As example let us consider $\mathbb{R}[X]$. We have two different types of irreducible polynomials. Of type (i) are the linear polynomials $X - \alpha$ (with a real zero α) and of type (ii) are the quadratic polynomials $X^2 + 2aX + b$ with pairs of conjugate complex zeros. The maximal ideals generated by the polynomials of type (i) correspond again to the geometric points of $\mathbb{R}$. There is no such relation for type (ii). In this case we have $V(X^2 + 2aX + b) = \emptyset$. Hence, there is no subvariety at all associated to this ideal. But if we calculate the coordinate ring $R(A)$ of this (not existing) subvariety A we obtain

$$R(A) = \mathbb{R}[X]/(X^2 + 2aX + b) \cong \mathbb{R} \oplus \mathbb{R}\bar{X}$$

with the relation $\bar{X}^2 = -2a\bar{X} - b$. In particular, $R(A)$ is a two-dimensional vector space. It is easy to show that $R(A)$ is isomorphic to $\mathbb{C}$. Instead of describing the "point" A as non-existing we should better describe it as a point of the real affine line which is $\mathbb{C}-$valued. (Recall that for the points of type (i) $R(A) \cong \mathbb{R}$.) This corresponds to the fact that the polynomial splits over the complex numbers $\mathbb{C}$ into two factors

$$(X + (a + \sqrt{a^2 - b}))(X + (a - \sqrt{a^2 - b})).$$

In this sense, the ideals of type (ii) correspond to conjugate pairs of complex numbers. Note that there is no way to distinguish between the two numbers from our point of view.

In the general situation for $\mathbb{K}$ one has to consider $\mathbb{L}-$valued points, where $\mathbb{L}$ is allowed to be any finite-dimensional field extension of $\mathbb{K}$.

3. Homomorphisms

Part 1. Let V and W be algebraic sets (not necessarily irreducible), resp.

$$R(V) = \mathbb{K}[X_1, X_2, \dots, X_n]/I, \qquad R(W) = \mathbb{K}[Y_1, Y_2, \dots, Y_m]/J$$

their coordinate rings. If $\Phi : V \to W$ is an arbitrary map and $f : W \to \mathbb{K}$ is a function then the pull-back $\Phi^*(f) := f \circ \Phi$ is a function $V \to \mathbb{K}$. If we interpret the elements of $R(W)$ as functions we want to call Φ an *algebraic map* if $\Phi^*(f) \in R(V)$ for every $f \in R(W)$. Roughly speaking this is equivalent to the fact that Φ "comes" from an algebra homomorphism $R(W) \to R(V)$. In this sense the coordinate rings are the dual objects to the algebraic varieties.

To make this precise, especially also to take care of the multiplicities, we should start from the other direction. Let $\Psi : R(W) \to R(V)$ be an algebra homomorphism. This homomorphism defines a homomorphism $\widetilde{\Psi}$ (where ν is the natural quotient map)

$$\widetilde{\Psi} = \Psi \circ \nu : \mathbb{K}[Y_1, Y_2, \dots, Y_m] \to R(V) \quad \text{with} \quad \widetilde{\Psi}(J) = 0 \mod I\ .$$

Such a homomorphism is given if we know the elements $\widetilde{\Psi}(Y_j)$. Conversely, if we fix elements $r_1, r_2, \dots, r_m \in R(V)$ then $\widetilde{\Psi}(Y_j) := r_j$, for $j = 1, \dots, m$ defines an algebra homomorphism $\widetilde{\Psi} : \mathbb{K}[Y_1, Y_2, \dots, Y_m] \to R(V)$. If $f(r_1, r_2, \dots, r_m) = 0 \mod I$ for all $f \in J$ then $\widetilde{\Psi}$ factorizes through $R(W)$. Such a map indeed defines a map Ψ^* on the set of geometric points,

$$\Psi^* : V \to W, \qquad \Psi^*(\alpha_1, \alpha_2, \dots .\alpha_n) := (\beta_1, \beta_2, \dots, \beta_m)$$

where the β_j are defined as

$$\beta_j = Y_j(\Psi^*(\alpha_1, \alpha_2, \dots, \alpha_n)) := \widetilde{\Psi}(Y_i)(\alpha_1, \alpha_2, \dots, \alpha_n).$$

We have to check whether $\Psi^*(\alpha) = \beta \in \mathbb{K}^m$ lies on the algebraic set W for $\alpha \in V$. For this we have to show that for all $f \in J$ we get $f(\Psi^*(\alpha)) = 0$ for $\alpha \in V$. But

$$f(\Psi^*(\alpha)) = f\big(Y_1(\Psi^*(\alpha)), \dots, Y_m(\Psi^*(\alpha))\big) = f\big(\widetilde{\Psi}(Y_1)(\alpha), \dots, \widetilde{\Psi}(Y_m)(\alpha)\big) = \widetilde{\Psi}(f)(\alpha)\ .$$

Now $\widetilde{\Psi}(f) = 0$, hence the claim.

Example 1. A function $V \to \mathbb{K}$ is given on the dual objects as a $\mathbb{K}-$algebra homomorphism

$$\Phi : \mathbb{K}[T] \to R(V) = \mathbb{K}[X_1, X_2, \dots, X_n]/I.$$

Such a Φ is uniquely given by choosing an arbitrary element $a \in R(V)$ and defining $\Phi(T) := a$. Here again you see the (now complete) interpretation of the elements of $R(V)$ as functions on V.

Example 2. The geometric process of choosing a (closed) point α on V can alternatively be described as giving a map from the algebraic variety consisting just of one point to the variety. Changing to the dual objects such a map is given as a map Φ_α from $R(V)$ to the field $\mathbb{K}$ which is the coordinate ring of a point. In this sense points correspond to homomorphisms of the coordinate ring to the base field $\mathbb{K}$. Such a homomorphism has of course a kernel $\ker \Phi_\alpha$ which is a maximal ideal. Again, it is the ideal defining the closed point α.

We will study this relation later. But first we take a different look on the situation.

Part 2. Let R be a $\mathbb{K}$-algebra where $\mathbb{K}$ is a field. The typical examples are the quotients of the polynomial ring $\mathbb{K}[X_1, X_2, \ldots, X_n]$. Let M be a module over R, i.e. a linear structure over R. In particular, M is a vector space over $\mathbb{K}$. Some standard examples of modules are obtained in the following manner. Let I be an ideal of R, $\nu : R \to R/I$ the quotient map then R/I is a module over R by defining $r \cdot \nu(m) := \nu(r \cdot m)$.

Definition. Let M be a module over R. The *annulator ideal* is defined to be

$$\operatorname{Ann}(M) := \{ r \in R \mid r \cdot m = 0, \ \forall m \in M \} .$$

That $\operatorname{Ann}(M)$ is an ideal is easy to check. It is also obvious that M is a module over $R/\operatorname{Ann}(M)$. By construction in the above example the ideal I is the annulator ideal of R/I. Hence, every ideal of R is the annulator ideal of a suitable $R-$module.

Definition. A module M is called a *simple* module if $M \neq \{0\}$ and M has only the trivial submodules $\{0\}$ and M.

Claim. *M is a simple module if and only if there is a maximal ideal P such that $M \cong R/P$.*

Proof. Note that the submodules of R/P correspond to the ideals lying between R and P. Hence, if P is maximal then R/P is simple. Conversely, given a simple module M take $m \in M, m \neq 0$. Then $R \cdot m$ is a submodule of M. Because $1 \cdot m = m$ the module $R \cdot m \neq \{0\}$, hence it is the whole module M. The map $\varphi(r) = r \cdot m$ defines a surjective map $\varphi : R \to M$. This map is an $R-$module map where R is considered as a module over itself. The kernel P of such a map is an $R-$submodule. But $R-$submodules of R are nothing else than ideals of R. In view of the next lecture where we drop the commutativity let us note already that submodules of a ring R are more precisely the left ideals of R. The kernel P has to be maximal otherwise the image of a maximal ideal lying between P and R would be a non-trivial submodule of M. Hence, $M \cong R/P$. $\square$

From this point of view the maximal ideals of $R(V)$ correspond to $R(V)-$module homomorphisms to simple $R(V)-$modules. If $R(V)$ is a algebra over the field $\mathbb{K}$, then a simple module M is of course a vector space over $\mathbb{K}$. By the above, we saw that it is even a field extension of $\mathbb{K}$. (Recall that $M \cong R/P$ with P a maximal ideal). Because $R(V)$ is finitely generated as $\mathbb{K}-$algebra it is a finite dimensional vector space over K (see [Ku,S.56]) hence, a finite (algebraic) field extension.

Observation. *The maximal ideals (the "points") of $R = \mathbb{K}[X_1, X_2, \ldots, X_n]/I$ correspond to the $\mathbb{K}-$algebra homomorphism from R to arbitrary finite (algebraic) field extensions $\mathbb{L}$ of the base field $\mathbb{K}$. We call these homomorphisms $\mathbb{L}-$valued points.*

In particular, if the field $\mathbb{K}$ is algebraically closed there are no nontrivial algebraic field extensions. Hence, there are only $\mathbb{K}-$valued points. If we consider reduced varieties (i.e. varieties whose coordinate rings are reduced rings) we get a complete dictionary. Let V be a variety, $P = I(V)$ the associated prime ideal generated as $P = (f_1, f_2, \ldots, f_r)$ with $f_i \in \mathbb{K}[X_1, X_2, \ldots, X_n]$ suitable polynomials, and $R(V)$ the coordinate ring R_n/P.
The points can be given in 3 ways:

(1) As classical points. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{K}^n$ with $f_1(\alpha) = f_2(\alpha) = \cdots = f_r(\alpha) = 0$.
(2) As maximal ideals in $R(V)$. They in turn can be identified with the maximal ideals in $\mathbb{K}[X_1, X_2, \ldots, X_n]$ which contain the prime ideal P. In an explicit manner these can be given as $(X_1 - \alpha_1,\ X_2 - \alpha_2,\ \ldots,\ X_n - \alpha_n)$ with the condition $f_1(\alpha) = f_2(\alpha) = \cdots = f_r(\alpha) = 0$.
(3) As surjective algebra homomorphisms $\phi : R(V) \to \mathbb{K}$. They are fixed by defining $\bar{X}_i \mapsto \phi(\bar{X}_i) = \alpha_i, i = 1, \ldots, n$ in such a way that $\phi(f_1) = \phi(f_2) = \cdots = \phi(f_r) = 0$.

The situation is different if we drop the assumption that $\mathbb{K}$ is algebraically closed. The typical changes can already be seen if we take the real numbers $\mathbb{R}$ and the real affine line. The associated coordinate ring is $\mathbb{R}[X]$. There are only two finite extension fields of $\mathbb{R}$, either $\mathbb{R}$ itself or the complex number field $\mathbb{C}$. If we consider $\mathbb{R}-$algebra homomorphism from $\mathbb{R}[X]$ to $\mathbb{C}$ then they are given by prescribing $X \mapsto \alpha \in \mathbb{C}$. If $\alpha \in \mathbb{R}$ we are again in the same situation as above (this gives us the type (i) maximal ideals). If $\alpha \notin \mathbb{R}$ then the kernel I of the map is a maximal ideal of type (ii) $I = (f)$ where f is a quadratic polynomial. f has α and $\bar{\alpha}$ as zeros. This says that the homomorphism $\Psi_{\bar{\alpha}} : X \mapsto \bar{\alpha}$ which is clearly different from $\Psi_\alpha : X \mapsto \alpha$ has the same kernel. In particular, for one maximal ideal of type (ii) we have two different homomorphisms. Note that the map $\alpha \to \bar{\alpha}$ is an element of the Galois group $G(\mathbb{C}/\mathbb{R}) = \{id, \tau\}$ where τ is complex conjugation. The two homomorphisms Ψ_α and $\Psi_{\bar{\alpha}}$ are related as $\Psi_{\bar{\alpha}} = \tau \circ \Psi_\alpha$.

This is indeed the general situation for $R(V)$, a finitely generated $\mathbb{K}-$algebra. In general, there is no 1-1 correspondence between (1) and (2) anymore. But there is a 1-1 correspondence between maximal ideals of $R(V)$ and orbits of $\mathbb{K}-$algebra homomorphism of $R(V)$ onto finite field extensions $\mathbb{L}$ of $\mathbb{K}$ under the action of the Galois group

$$G(\mathbb{L}/\mathbb{K}) := \{\, \sigma : \mathbb{L} \to \mathbb{L} \ \text{ an automorphism of fields with } \sigma_{|\mathbb{K}} = id\} \ .$$

4. Some Comments on the noncommutative situation

For the following let R be a (not necessarily commutative) algebra over the field $\mathbb{K}$. First, we have to distinguish in this more general context left ideals (e.g. subrings I which are invariant under multiplication with R from the left), right ideals and two-sided ideals (which are left and right ideals). To construct quotient rings two-sided ideals are needed. If we use the term ideal without any additional comment we assume the ideal to be a two-sided one.

We want to introduce the concepts of prime ideals, maximal ideals, etc.. A first definition of a prime ideal could be as follows. We call a two-sided ideal I *prime* if the quotient R/I contains no zero-divisor. This definition has the drawback that there are rings without any prime ideal at all. Take for example the ring of 2×2 matrices. Beside the ideal $\{0\}$ and the whole ring the matrix ring does not contain any other ideal. To see this assume there is an ideal I which contains a non-zero matrix A. By applying elementary operations from the left and the right we can transform any matrix to normal form which is a diagonal matrix with just 1 (at least one) and 0 on the diagonal. By multiplication with a permutation matrix we can achieve any pattern in the diagonal. These operations keep us inside the ideal. Adding suitable elements we see that the unit matrix is in the ideal. Hence the ideal is the whole ring. But obviously, the matrix ring has zero divisors. Hence, $\{0\}$ is not prime in this definition. We see that this ring does not contain any prime ideal at all with respect to the definition. We choose another name for such ideals: they are called *complete prime ideals*.

Definition. A (two-sided) ideal I is called a *prime ideal* if for any two ideals J_1 and J_2 with $J_1 \cdot J_2 \subseteq I$ it follows that $J_1 \subseteq I$ or $J_2 \subseteq I$.

This definition is equivalent to the following one.

Definition. A (two-sided) ideal I is called a *prime ideal* if for any two elements $a, b \in R$ with $a \cdot R \cdot b \subseteq I$ it follows that $a \in I$ or $b \in I$.

Proof. (2. D) $\implies$ (1. D): Take $J_1 \not\subseteq I$ and $J_2 \not\subseteq I$ ideals. We have to show that $J_1 \cdot J_2 \not\subseteq I$. For this choose $x \in J_1 \setminus I$ and $y \in J_2 \setminus I$. Then $x \cdot R \cdot y \subseteq J_1 \cdot J_2$ but there must be some $r \in R$ such that $x \cdot r \cdot y \notin I$ due to the condition that I is prime with respect to (2. D). Hence, $J_1 \cdot J_2 \not\subseteq I$ which is the claim.
(1. D) $\implies$ (2. D): Take $a, b \in R$. The ideals generated by these elements are RaR and RbR. The product of these "principal" ideals is not a principal ideal anymore. It is $RaR \cdot RbR = RaRbR := (arb \mid r \in R)$. Assume $arb \in I$ for all $r \in R$. Hence $(RaR)(RbR) \subseteq I$ and because I is prime we obtain by the first definition (1. D) that either RaR or RbR are in I. Taking as element of R the 1 we get $a \in R$ or $b \in R$. $\square$

Every ideal which is a complete prime is prime. Obviously, the condition (2. D) is a weaker condition than the condition that already from $a \cdot b \in I$ it follows that $a \in R$ or $b \in R$ (which is equivalent to: R/I contains no zero-divisors). If R is commutative then they coincide. In this case $a \cdot r \cdot b = r \cdot a \cdot b$, and with $a \cdot b \in I$ also $r \cdot a \cdot b \in I$ which is no additional condition. Here you see clearly where the noncommutativity enters the picture. In the ring of matrices the ideal $\{0\}$ is prime because if after fixing two matrices A and B we obtain $A \cdot T \cdot B = 0$ for any matrix T then either A or B has to be the zero matrix. This shows that the zero ideal in the matrix ring is a prime ideal.

Maximal ideals are defined again as in the commutative setting just as maximal elements in the (non-empty) set of ideals. By Zorn's lemma there exist maximal ideals.

Claim. *If M is a maximal ideal then it is a prime ideal.*

Proof. Take I and J ideals of R which are not contained in M. Then by the maximality of M we get $(I + M) = R$ and $(J + M) = R$ hence,

$$R \cdot R = R = (I + M)(J + M) = I \cdot J + M \cdot J + I \cdot M + M \cdot M \ .$$

If we assume $I \cdot J \subseteq M$ then $R \subseteq M$ which is a contradiction. Hence $I \cdot J \not\subseteq M$. This shows M is prime. $\square$

By this result we see that every ring has prime ideals.
In the commutative case if we approach the theory of ideals from the point of view of modules over R we obtain an equivalent description. This is not true anymore in the noncommutative setting. For this let M be a (left-)module over R. As above we define

$$\mathrm{Ann}(M) := \{\, r \in R \mid r \cdot m = 0, \forall m \in M \} \ ,$$

the annulator of the module M. $\mathrm{Ann}(M)$ is a two-sided ideal. Clearly, it is closed under addition and is a left ideal. (This is even true for an annulator of a single element $m \in M$). It is also a right ideal: let $s \in \mathrm{Ann}(M)$ and $t \in R$ then $(st)m = s(tm) = 0$ because s annulates also tm.

Definition. An ideal I is called a *primitive ideal* if I is the annulator ideal of a simple module M.

Let us call the set of prime, resp. primitive, resp. maximal ideals $\operatorname{Spec}(R)$, $\operatorname{Priv}(R)$ and $\operatorname{Max}(R)$.

Claim.

$$\operatorname{Spec}(R) \quad \supseteq \quad \operatorname{Priv}(R) \quad \supseteq \quad \operatorname{Max}(R) \ .$$

Proof. (1). Let P be a maximal ideal. Then R/P is a (left-)module. Unfortunately, it is not necessarily simple (as module). The submodules correspond to left-ideals lying between P and R. Choose Q a maximal left ideal lying above P. Then R/Q is a simple (left-) module and $P \cdot (R/Q) = 0$ because $P \cdot R = P \subseteq Q$. Hence, $P \subseteq \operatorname{Ann}(R/Q)$ and because $\operatorname{Ann}(R/Q)$ is a two-sided ideal we get equality.
(2). Take $P = \operatorname{Ann}(M)$, a primitive ideal. Assume P is not prime. Then there exist $a, b \in R$ but $a, b \notin P$ such that for all $r \in R$ we get $arb \in P$. This implies $arbm = 0$ for all $m \in M$ but $bm \neq 0$ for at least one m. Now $B = R(bm)$ is a non-vanishing submodule. Obviously, $a \in \operatorname{Ann}(B)$, hence $B \neq M$. This contradicts the simplicity of M. $\square$

Clearly, in the commutative case $\operatorname{Priv}(R) = \operatorname{Max}(R)$. Let me just give an example from [GoWa] that in the noncommutative case they fall apart. Take V an infinite-dimensional $\mathbb{C}-$vector space. Let R be the algebra of linear endomorphisms of V and I the nontrivial two-sided ideal consisting of linear endomorphisms with finite-dimensional image. The vector space V is an $R-$module by the natural action of the endomorphisms. We get that $V = R \cdot v$ where v is any non-zero vector of V. This implies that the module V is simple and that $\operatorname{Ann}(V) = \{0\}$. Hence $\{0\}$ is primitive, but it is not maximal because I is lying above it.

In the commutative case we saw that we could interpret homomorphisms of the coordinate ring (which is an algebra if we consider varieties over a base field) into a field as points of the associated space. Indeed, it is possible to give such an interpretation also in the noncommutative setting. Let me give an example, for details see [Ma-1]. Let $M_q(2)$ for $q \in \mathbb{C}, q \neq 0$ be the (noncommutative) $\mathbb{C}-$algebra generated by a, b, c, d, subject to the relations:

$$\begin{gathered} ab = \frac{1}{q}\, ba, \qquad ac = \frac{1}{q}\, ca, \qquad ad = da + \left(\frac{1}{q} - q\right) bc, \\ bc = cb, \qquad bd = \frac{1}{q}\, db, \qquad cd = \frac{1}{q}\, dc \ . \end{gathered} \tag{4-1}$$

This algebra is constructed by first considering all possible words in a, b, c, d. This defines the free noncommutative algebra of this alphabet. Multiplication is defined by concatenation of the words. Take the ideal generated by the expressions (left-side) − (right-side) of all the relations (4-1) and build the quotient algebra. Note that for $q = 1$ we obtain the commutative algebra of polynomial functions on the space of all 2×2 matrices over $\mathbb{C}$. In this sense the algebra $M_q(2)$ represents the "quantum matrices" as a "deformation of the usual matrices". To end up with the *quantum group* $Gl_q(2)$ we would have to add another element for the formal inverse of the quantum determinant $D = ad - \dfrac{1}{q}\, bc$.[3]

Now let A be another algebra. We call a $\mathbb{C}-$linear algebra homomorphism $\Psi \in \mathrm{Hom}(M_q(2), A)$ an *$A-$valued point* of $M_q(2)$. It is called a *generic point* if Ψ is injective. Saying that a linear map Ψ is an algebra homomorphism is equivalent to saying that the elements $\Psi(a), \Psi(b), \Psi(c), \Psi(d)$ fulfill the same relations (4-1) as the a, b, c and d. One might interpret Ψ as a point of the "quantum group". But be careful, it is only possible to "multiply" the two matrices if the images of the two maps

$$\Psi_1 \sim B_1 := \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \qquad \Psi_2 \sim B_2 := \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

lie in a common algebra A_3, i.e. $a_1, b_1, c_1, d_1 \in A_1 \subseteq A_3$ and $a_2, b_2, c_2, d_2 \in A_2 \subseteq A_3$. Then we can multiply the two matrices $B_1 \cdot B_2$ as prescribed by the usual matrix product and obtain another matrix B_3 with coefficients $a_3, b_3, c_3, d_3 \in A_3$. This matrix defines only then a homomorphism of $M_q(2)$, i.e. an A_3-valued point if $\Psi_1(M_q(2))$ commutes with $\Psi_2(M_q(2))$ as subalgebras of A_3. In particular, the product of Ψ with itself is not an $A-$valued point of $M_q(2)$ anymore. One can show that it is an $A-$valued point of $M_{q^2}(2)$.

Because in the audience there a couple people who had and still have their share in developing the fundamentals of quantum groups (the Wess-Zumino approach) there is no need to give a lot of references on the subject. Certainly, these people know it much better than I do. For the reader let me just quote one article by Julius Wess and Bruno Zumino [WZ] where one finds references for further study in this direction. Let me only give the following three references of books, resp. papers of Manin which are more connected to the theme of these lectures: "Quantum groups and noncommutative geometry" [Ma-1], "Topics in noncommutative geometry" [Ma-2], and "Notes on quantum groups and the quantum de Rham complexes" [Ma-3].

For the general noncommutative situation I like to recommend Goodearl and Warfield, "An introduction to noncommutative noetherian rings" [GoWa] and Borho, Gabriel, Rentschler, "Primideale in Einhüllenden auflösbarer Liealgebren" [BGR]. These books are still completely on the algebraic side of the theory. For the algebraic geometric side there is still not very much available. Unfortunately, I am also not completely aware of the very recent developments of the theory. The reader may use the two articles [Ar-2] and [R] as starting points for his own exploration of the subject.

[3] There are other objects which carry also the name quantum groups.

5. Affine schemes

Returning to the commutative setting let R be again a commutative ring with unit 1. We do not assume R to be an algebra over a field $\mathbb{K}$. If we consider the theory of differentiable manifolds the model manifold is $\mathbb{R}^n$. Locally any arbitrary manifold looks like the model manifold. Affine schemes are the "model spaces" of algebraic geometry. General schemes will locally look like affine schemes. Contrary to the differentiable setting, there is not just one model space but a lot of them. Affine schemes are very useful generalizations of affine varieties. Starting from affine varieties V over a field $\mathbb{K}$ we saw that we were able to assign dual objects to them, the coordinate rings $R(V)$. The geometric structure of V (subvarieties, points, maps, ...) are represented by the algebraic structure of $R(V)$ (prime ideals, maximal ideals, ring homomorphisms, ...). After dualization we are even able to extend our notion of "space" in the sense that we can consider more general rings and regard them as dual objects of some generalized "spaces". In noncommutative quantum geometry one even studies certain noncommutative algebras over a field $\mathbb{K}$. Quantum spaces are the dual objects of these algebras. We will restrict ourselves to the commutative case, but we will allow arbitrary rings.

What are the dual objects (dual to the rings) which generalize the concept of a variety. We saw already that prime ideals of the coordinate ring correspond to subvarieties and that closed prime ideals (at least if the field $\mathbb{K}$ is algebraically closed) correspond to points. It is quite natural to take as space the set $\operatorname{Spec}(R)$ together with its Zariski topology. But this is not enough. If we take for example $R_1 = \mathbb{K}$ and $R_2 = \mathbb{K}[\epsilon]/(\epsilon^2)$ then in both cases $\operatorname{Spec}(R_i)$ consists just of one point. It is represented in the first case by the ideal $\{0\}$ in the second case by (ϵ). Obviously, both Spec coincide. Let us compare this with the differentiable setting. For an arbitrary differentiable manifold the structure is not yet given if we consider the manifold just as a topological manifold. We can fix its differentiable structure if we tell what the differentiable functions are. The same is necessary in the algebraic situation. Hence, $\operatorname{Spec}(R)$ together with the functions (which in the case of varieties correspond to the elements of R) should be considered as "space". So the space associated to a ring R should be $(\operatorname{Spec}(R), R)$. In fact, $\operatorname{Spec}(R)$ is not a data independent of R. Nevertheless, we will write both information in view of globalizations of the notion. Compare this again with the differentiable situation. If you have a manifold which is $\mathbb{R}^n$ (the model manifold) then the topology is fixed. But if you have an arbitrary differentiable manifold then you need a topology at the first place to define coordinate charts at all. In view of these globalizations we additionally have to replace the ring of functions by a data which will give us all local and global functions together. Note that in the case of compact complex analytic manifolds there would exist no non-constant analytic functions at all. The right setting for this is the language of sheaves. Here it is not the time and place to introduce this language. Just let me give you a very rough idea. A sheaf is the coding of an object which is local and global in a compatible way. A standard example (which is in some sense too simple) is the sheaf of differentiable functions on a differentiable manifold X. It assigns to every open set U the ring of differentiable functions defined on U. The compatibility just means that this assignment is compatible with the restriction of the sets where the functions are defined on. In Appendix A to this lecture you will find the exact definition of a sheaf of rings. So, given a ring R its associated affine scheme is the pair $(\mathrm{Spec}(R), \mathcal{O}_R)$ where $\mathrm{Spec}(R)$ is the set of prime ideals made into a topological space by the Zariski topology and $\mathcal{O}_R$ is a sheaf of rings on $\mathrm{Spec}(R)$ which we will define in a minute. For simplicity this pair is sometimes just called $\mathrm{Spec}(R)$.

Recall that the sets $V(S) := \{ P \in \mathrm{Spec}(R) \mid P \supseteq S\}$, where S is any $S \subseteq R$, are the closed sets. Hence the sets $\mathrm{Spec}(R) \setminus V(S)$ are exactly the open sets of $X := \mathrm{Spec}(R)$. There are some special open sets in X. For a single element $f \in R$ we define

$$X_f := \mathrm{Spec}(R) \setminus V(f) = \{ P \in \mathrm{Spec}(R) \mid f \notin P\} \ . \tag{5-1}$$

The set $\{X_f, f \in R\}$ is a basis of the topology which says that every open set is a union of X_f. This is especially useful because the X_f are again affine schemes. More precisely, $X_f = \mathrm{Spec}(R_f)$. Here the ring R_f is defined as the ring of fractions with the powers of f as denominators:

$$R_f := \{ \frac{g}{f^n} \mid g \in R,\ n \in \mathbb{N}_0 \} \ .$$

Let me explain this construction. It is a generalization of the way how one constructs the rational numbers from the integers. For this let S be a multiplicative system, i.e. a subset of R which is multiplicatively closed and contains 1. (In our example, $S := \{1, f, f^2, f^3, \dots\}$.) Now introduce on the set of pairs in $R \times S$ the equivalence relation

$$(t, s) \sim (t', s') \iff \exists s'' \in S \text{ such that } s''(s't - st') = 0 \ .$$

The equivalence class of (s, t) is denoted by $\frac{s}{t}$. There is always a map $R \to R_f$ given by $r \mapsto \frac{r}{1}$. The ideals in R_f are obtained by mapping the ideals I of R to R_f and multiplying them by R_f : $R_f \cdot I$. By construction, f is a unit in R_f. Hence, if $f \in P$ where P is a prime ideal then $R_f = R_f \cdot P$. If $f \notin P$ then $R_f \cdot P$ still is a prime ideal of R_f. This shows $X_f = \mathrm{Spec}(R_f)$. For details see [Ku].

You might ask what happens if f is nilpotent, i.e. if there is a $n \in \mathbb{N}$ such that $f^n = 0$. In this case f is contained in any prime ideal of R_f. Hence $\mathrm{Spec}(R_f) = \emptyset$ in agreement with $R_f = \{0\}$.

If f is not a zero divisor the map $R \to R_f$ is an embedding and if f is not a unit in R the ring R_f will be bigger. This is completely in accordance with our understanding of R resp. R_f as functions on X, resp. on the honest subset X_f. Passing from X to X_f

is something like passing from the global to the more local situation. This explains why this process of taking the ring of fractions with respect to some multiplicative subset S is sometimes called localization of the ring. The reader is adviced to consider the following example. Let P be a prime ideal, show that $S = R \setminus P$ is a multiplicative set. How can one interpret the ring of fractions of R with respect to S?

Now we define our sheaf $\mathcal{O}_R$ for the basis sets X_f. In $X_f \cap X_g$ are the prime ideals which neither contain f nor g. Hence they do not contain $f \cdot g$. It follows that $X_f \cap X_g = X_{fg}$. We see that the set of the X_f are closed under intersections. Note also that $X_1 = X$ and $X_0 = \emptyset$. We define

$$\mathcal{O}_R(X) := R, \qquad \mathcal{O}_R(X_f) := R_f \ . \tag{5-2}$$

For $X_{fg} = X_f \cap X_g \subseteq X_f$ we define the restriction map

$$\rho^f_{fg} : R_f \to (R_f)_g = R_{fg}, \quad r \mapsto \frac{r}{1} \ .$$

It is easy to check that all the maps $\rho^{\cdot}_{\cdot}$ are compatible on the intersections of the basis open sets. In Appendix B I will show that the other sheaf axioms are fulfilled for the X_f with respect to their intersections. Hence, we have defined the sheaf $\mathcal{O}_R$ on a basis of the topology which is closed under intersections. The whole sheaf is now defined by some general construction. We set

$$\mathcal{O}_R(U) := \operatorname*{proj\,lim}_{X_f \subseteq U} \mathcal{O}_R(X_f)$$

for a general open set. For more details see [EH]. Let us collect the facts.

Definition. Let R be a commutative ring. The pair $(\mathrm{Spec}(R), \mathcal{O}_R)$, where $\mathrm{Spec}(R)$ is the space of prime ideals with the Zariski topology and $\mathcal{O}_R$ is the sheaf of rings on $\mathrm{Spec}(R)$ introduced above is called the *associated affine scheme* $\mathrm{Spec}(R)$ of R. The sheaf $\mathcal{O}_R$ is called the *structure sheaf* of $\mathrm{Spec}(R)$.

Let me explain in which sense the elements f of an arbitrary ring R can be considered as functions, i.e. as prescriptions how to assign a value from a field to every point. This gives me the opportunity to introduce another important concept which is related to points: the residue fields. Fix an element $f \in R$. Let $[P] \in \mathrm{Spec}(R)$ be a (not necessarily closed) point, i.e. P is a prime ideal. We define

$$f([P]) := f \mod P \in R/P$$

in a first step. From the primeness of P it follows that R/P is an integral domain ring (i.e. it contains no zero-divisor). Hence $S := (R/P) \setminus \{0\}$ is a multiplicative system and the ring of fractions, denoted by $\mathrm{Quot}(R/P)$, is a field, the quotient field. Because R/P is an integral domain it can be embedded into its quotient field. Hence, $f([P])$ is indeed an element of a field. Contrary to the classical situation, if we change the point $[P]$ the field $\mathrm{Quot}(R/P)$ will change too.

Example 1. Take again $R = \mathbb{C}[X, Y]$ and $f \in R$. Here we have three different types of points in $\mathrm{Spec}(R)$.
Type (i): the closed points $[M]$ with $M = (X - \alpha, Y - \beta)$ a maximal ideal. We write $f = f_0 + (X - \alpha) \cdot g + (Y - \beta) \cdot h$ with $f_0 = f(\alpha, \beta) \in \mathbb{C}$ and $g, h \in R$. Now

$$f([M]) = f \mod M = f_0 + (X - \alpha) \cdot g + (Y - \beta) \cdot h \mod M = f_0 \ .$$

The quotient R/M is already a field, hence it is the residue field. In our case it is even the base field $\mathbb{C}$. The value $f([M])$ is just the value we obtain by plugging the point (α, β) into the polynomial f. Note that the points are subvarieties of dimension 0.
Type (ii): the points $[P]$ with $P = (h)$, a principal ideal. Here h is an irreducible polynomial in the variables X and Y. If we calculate R/P we obtain $\mathbb{C}[X, Y]/(h)$ which is not a field. As residue field we obtain $\mathbb{C}(X, Y)/(h)$. This field consists of all rational expressions in the variables X and Y with the relation $h(X, Y) = 0$. This implies that the transcendence degree of the residue field over the base field is one, i.e. one of the variables X or Y is algebraically independent over $\mathbb{C}$ and the second variable is in an algebraic relation with the first and the elements of $\mathbb{C}$. Note that the coordinate ring has (Krull-) dimension one and the subvariety corresponding to $[P]$ is a curve, i.e. is an object of geometric dimension one.
Type (iii): $[\{0\}]$ the zero ideal. In this case $R/P = \mathbb{C}[X, Y]$ and the residue field is $\mathbb{C}(X, Y)$ the rational function field in two variables. In particular, its transcendence degree is two and coincides with the (Krull-)dimensions of the coordinate ring and the geometric dimension of the variety $V(\{0\})$ which equals the whole affine plane $\mathbb{C}^2$.

Strictly speaking, we have not shown (and will not do it here) that there are no other prime ideals. But this is in fact true, see [Ku]. The equality of the transcendence degree of the residue field and the (Krull-) dimension of the coordinate ring obtained above is true for all varieties over arbitrary fields. For example, if we replace $\mathbb{C}$ by $\mathbb{R}$ we obtain for the closed points, the maximal ideals, either $\mathbb{R}$ or $\mathbb{C}$ as residue fields. Both fields have transcendence degree 0 over $\mathbb{R}$.

Example 2. Consider $R = \mathbb{Z}$, the integers, then $\mathrm{Spec}(\mathbb{Z})$ consists of the zero ideal and the principal ideals generated by prime numbers. As residue field we obtain for $[0]$ the field $\mathrm{Quot}(\mathbb{Z}/(0)) = \mathbb{Q}$ and for the point $[(p)]$ (which is a closed point) $\mathbb{F}_p = \mathbb{Z}/(p)$, the prime field of characteristic p. In particular, we see at this example that even for the maximal points the residue field can vary in an essential way. Note that $\mathbb{Z}$ is not an algebra over a fixed base field.

Up to now we considered one ring, resp. one scheme. In any category of objects one has maps between the objects. Let $\Phi : R \to S$ be a ring homomorphism. If I is any ideal of S, then $\Phi^{-1}(I)$ is an ideal of R. The reader is advised to check that if P is prime then $\Phi^{-1}(P)$ is again prime. Hence, $\Phi^* : P \mapsto \Phi^{-1}(P)$ is a well-defined map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$. Indeed, it is even continuous because the pre-image of a closed set is again closed. Let $X = (\operatorname{Spec}(S), \mathcal{O}_S)$ and $Y = (\operatorname{Spec}(R), \mathcal{O}_R)$ be two affine schemes. The map Φ induces also a map on the level of the structure sheaves $\Phi_* : \mathcal{O}_R \to \mathcal{O}_S$. The pair (Φ^*, Φ_*) of maps fulfills certain compatibility conditions which makes them to a homomorphism of schemes.

We will not work with schemes in general later on but let me give at least for completeness the definition here.

Definition. (a) A *scheme* is a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf $\mathcal{O}_X$ of rings on X, such that X is locally isomorphic to affine schemes $(\operatorname{Spec}(R), \mathcal{O}_R)$. This says that for every point $x \in X$ there is an open set U containing x, and a ring R (it may depend on the point x) such that the affine scheme $(\operatorname{Spec}(R), \mathcal{O}_R)$ is isomorphic to the scheme $(U, \mathcal{O}_{X|U})$. In other words there is a homeomorphism $\Psi : U \to \operatorname{Spec}(R)$ such that there is an isomorphism of sheaves

$$\Psi^{\#} : \mathcal{O}_R \cong \Psi_*(\mathcal{O}_{X\,|U}) \ .$$

Here the sheaf $\Psi_*(\mathcal{O}_{X\,|U})$ is defined to be the sheaf on $\operatorname{Spec}(R)$ given by the assignment

$$\Psi_*(\mathcal{O}_{X\,|U})(W) := \mathcal{O}_X(\Psi^{-1}(W)), \qquad \text{for every open set } W \subseteq \operatorname{Spec}(R).$$

(b) A scheme is called an *affine scheme* if it is globally isomorphic to an affine scheme $(\operatorname{Spec}(R), \mathcal{O}_R)$ associated to a ring R.

Fact. *The category of affine schemes is equivalent to the category of commutative rings with unit with the arrows (representing the maps) reversed.*

There are other important concepts in this theory. First, there is the concept of a scheme over another scheme. This is the right context to describe families of schemes. Only within this framework it is possible to make such useful things precise as degenerations, moduli spaces etc. Note that every affine scheme is in a natural way a scheme over $\operatorname{Spec}(\mathbb{Z})$, because for every ring R we have the natural map $\mathbb{Z} \to R,\ n \mapsto n \cdot 1$. Taking the dual map introduced above we obtain a homomorphism of schemes.
If R is a $\mathbb{K}-$algebra with $\mathbb{K}$ a field then we have the map $\mathbb{K} \to R,\ \alpha \mapsto \alpha \cdot 1$, which is a ring homomorphism. Hence, we always obtain a map: $\operatorname{Spec}(R) \to \operatorname{Spec}(\mathbb{K}) = (\{0\}, K)$. By considering the coordinate ring $R(V)$ of an affine variety V over a fixed algebraically closed field $\mathbb{K}$ and assigning to it the affine scheme $\operatorname{Spec}(R(V))$ we obtain a functor from the category of varieties over $\mathbb{K}$ to the category of schemes over $\mathbb{K}$. The schemes

corresponding to the varieties are the *irreducible and reduced noetherian affine schemes of finite type over* $\mathrm{Spec}(\mathbb{K})$. The additional properties of the scheme are nothing else as the corresponding properties for the defining ring $R(V)$. Here finite type means that $R(V)$ is a finitely generated $\mathbb{K}-$algebra. You see again in which sense the schemes extend our geometric objects from the varieties to more general "spaces".

The second concept is the concept of a *functor of points* of a scheme. We saw already at several places in the lectures that points of a geometric object can be described as homomorphisms of the dual (algebraic) object into some simple (algebraic) object. If X is a scheme we can associate to it the following functor from the category of schemes to the category of sets: $h_X(S) = \mathrm{Hom}(S, X)$. Here S is allowed to be any scheme and $\mathrm{Hom}(S, X)$ is the set of homomorphisms of schemes from S to the fixed scheme X. Such a homomorphism is called an *$S-$valued point* of X. Note that we are in the geometric category, hence the order of the elements in $\mathrm{Hom}(.,.)$ is just the other way round compared to the former lectures. The functor h_X is called the *functor of points* associated to X. Now X is completely fixed by the functor h_X. In categorical language: X represents its own functor of points. The advantage of this view-point is that certain questions of algebraic geometry, like the existence of a moduli space for certain geometric data, can be easily transfered to the language of functors. One can extract already a lot of geometric data without knowing whether there is indeed a scheme having this functor as functor of points (i.e. representing the functor). If you want to know more about this beautiful subject you should consult [EH] and [Mu-2].

Appendix A: The definition of a sheaf of rings. A *presheaf* $\mathcal{F}$ of rings over a topological space X assigns to every open set U in X a ring $\mathcal{F}(U)$ and to every pair of open sets $V \subseteq U$ a homomorphism of rings

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V),$$

(the so called *restriction map*) in such a way that

$$\rho_U^U = id,$$
$$\rho_V^U \circ \rho_U^W = \rho_V^W \quad \text{for } V \subseteq U \subseteq W .$$

Instead of $\rho_V^U(f)$ for $f \in \mathcal{F}(U)$ we often use the simpler notation $f_{|V}$. A presheaf is called a *sheaf* if for every open set U and every covering (U_i) of this open set we have in addition:

(1) if $f, g \in \mathcal{F}(U)$ with

$$f_{|U_i} = g_{|U_i}$$

for all U_i then $f = g$,

(2) if a set of $f_i \in \mathcal{F}(U_i)$ is given with

$$f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$$

then there exists a $f \in \mathcal{F}(U)$ with
$$f_{|U_i} = f_i.$$
Given two sheaves of rings $\mathcal{F}$ and $\mathcal{G}$ on X. By a *sheaf homomorphism*
$$\psi : \mathcal{F} \to \mathcal{G}$$
we understand an assignment of a ring homomorphism ψ_U (for every open set U)
$$\psi_U : \mathcal{F}(U) \to \mathcal{G}(U),$$
which is compatible with the restriction homomorphisms
$$\begin{array}{ccccc} U & \qquad & \mathcal{F}(U) & \xrightarrow{\psi_U} & \mathcal{G}(U) \\ \bigcup & & \downarrow & & \downarrow \\ V & & \mathcal{F}(V) & \xrightarrow{\psi_V} & \mathcal{G}(V) \end{array}$$
More information you find in [Sch].

Appendix B. The structure sheaf $\mathcal{O}_R$. In this appendix I like to show that the sheaf axioms for the structure sheaf $\mathcal{O}_R$ on $X = \mathrm{Spec}(R)$ are fulfilled if we consider only the basis open sets $X_f = \mathrm{Spec}(R) \setminus V(f)$. Recall that the intersection of two basis basis open sets $X_f \cap X_g = X_{fg}$ is again a basis open set. The sheaf $\mathcal{O}_R$ on the basis open sets was defined to be $\mathcal{O}_R(X_f) = R_f$ and the restriction maps were the natural maps
$$R_f \;\to\; (R_f)_g \;=\; R_{fg}, \qquad r \mapsto \frac{r}{1}\ .$$
Here I am following very closely the presentation in [EH].

Lemma 1. *The set $\{X_f \mid f \in R\}$ is a basis of the topology.*

Proof. We have to show that every open set U is a union of such X_f. By definition,
$$U = \mathrm{Spec}(R) \setminus V(S) = \mathrm{Spec}(R) \setminus (\bigcap_{f\in S} V(f)) = \bigcup_{f\in S}(\mathrm{Spec}(R) \setminus V(f)) = \bigcup_{f\in S} X_f\ . \quad \square$$

Obviously, only a set of generators $\{f_i \mid i \in J\}$ of the ideal generated by the set S is needed. Hence, if R is a noetherian ring every open set can already be covered by finitely manx X_f.

Lemma 2. *Let* $X = \mathrm{Spec}(R)$ *and* $\{f_i\}_{i\in J}$ *a set of elements of* R *then the union of the sets* X_{f_i} *equals* X *if and only if the ideal generated by the* f_i *equals the whole ring* R.

Proof. The union of the X_{f_i} covers $\mathrm{Spec}(R)$ iff no prime ideal of R contains all the f_i. But every ideal strictly smaller than the whole ring is dominated by a maximal (and hence prime) ideal. The above can only be the case iff the ideal generated by the f_i is the whole ring. □

Lemma 3. *The affine scheme* $X = \mathrm{Spec}(R)$ *is a quasicompact space. This says every open cover of* X *has a finite subcover.*

Proof. Let $X = \bigcup_{j\in J} X_j$ be a cover of X. Because the basis open set X_f are a basis of the topology, every X_j can be given as union of X_{f_i}. Altogether, we get a refinement of the cover $X = \bigcup_{i\in I} X_{f_i}$. By Lemma 2 the ideal generated by these f_i is the whole ring. In particular, 1 is a finite linear combination of the f_i. Taking only these f_i which occur with a non-zero coefficient in the linear combination we get (using Lemma 2 again) that $X_{f_{i_k}}$, $k = 1, .., r$ is a finite subcover of X. Taking for every k just one element X_{j_k} containing $X_{f_{i_k}}$ we obtain a finite number of sets which is a subcover from the cover we started with. □

Note that this space is not called a compact space because the Hausdorff condition that every distinct two points have disjoint open neighbourhoods is obviously not fulfilled.

The following proposition says that the sheaf axioms (1) and (2) from App. A for the basis open sets are fulfilled.

Proposition. *Let* X_f *be coverd by* $\{X_{f_i}\}_{i\in I}$.

(a) Let $g, h \in R_f = \mathcal{O}_R(X_f)$ *with* $g = h$ *as elements in* $R_{f_i} = \mathcal{O}_R(X_{f_i})$ *for every* $i \in I$*, then* $g = h$ *also in* R_f.

(b) Let $g_i \in R_{f_i}$ *be given for all* $i \in I$ *with* $g_i = g_j$ *in* $R_{f_i f_j}$ *, then there exist a* $g \in R_f$ *with* $g = g_i$ *in* R_{f_i}.

Proof. Because $X_f = \mathrm{Spec}(R_f)$ is again an affine scheme it is enough to show the proposition for $R_f = R$, where R is an arbitrary ring. Let $X = \bigcup_{i\in I} X_{f_i}$.

(a) Let $g, h \in R$ be such that they map to the same element in R_{f_i}. This can only be the case if in R we have

$$f_i^{n_i} \cdot (g - h) = 0, \qquad \forall i \in I,$$

(see the construction of the ring of fractions above). Due to the quasicompactness it is enough to consider finitely many f_i, $i = 1, .., r$. Hence, there is a N such that for every i the element f_i^N annulates $(g-h)$. There is another number M, depending on N and r, such that we have for the following ideals

$$(f_1^N, f_2^N, \dots, f_r^N) \supseteq (f_1, f_2, \dots, f_r)^M \ .$$

Because the X_{f_i}, $i = 1, .., r$ are a cover of X the ideal on the right side equals (1). Hence, also the ideal on the left. Combining 1 as linear combination of the generator we get

$$1 \cdot (g-h) = (c_1 f_1^N + c_2 f_2^N \cdots + c_r f_r^N)(g-h) = 0 \ .$$

This shows (a)
(b) Let $g_i \in R_{f_i}$, $i \in I$ be given such that $g_i = g_j$ in $R_{f_i f_j}$. This says there as a N such that

$$(f_i f_j)^N g_i = (f_i f_j)^N g_j$$

in R. Note that every g_i can be written as $\dfrac{g_i^*}{f_i^{k_i}}$ with $g_i^* \in R$. Hence, if N is big enough the elements $f_i^N g_i$ are in R. Again by the quasicompactness a common N will do it for every pair (i,j). Using the same arguments as in (a) we get

$$1 \ = \ \sum e_i f_i^N, \qquad e_i \in R \ .$$

This formula corresponds to a "partition of unity". We set

$$g \ = \ \sum e_i f_i^N g_i \ .$$

We get

$$f_j^N g = \sum_i f_j^N e_i f_i^N g_i = \sum_i e_i f_i^N f_j^N g_j = f_j^N g_j \ .$$

This shows $g = g_j$ in R_{f_j}. $\square$

6. Examples of Schemes

1. Projective Varieties. Affine Varieties are examples of affine schemes over a field $\mathbb{K}$. They have been covered thoroughly in the other lectures. For completeness let me mention that it is possible to introduce the *projective space* $\mathbb{P}^n_{\mathbb{K}}$ of dimension n over a field $\mathbb{K}$. It can be given as orbit space $(\mathbb{K}^{n+1} \setminus \{0\})/\sim$, where two $(n+1)-$tuple α and β are equivalent if $\alpha = \lambda \cdot \beta$ with $\lambda \in \mathbb{K}$, $\lambda \neq 0$. *Projective varieties* are defined to be the vanishing sets of homogeneous polynomials in $n+1$ variables. See for example [Sch] for more information. What makes them so interesting is that they are compact varieties (if $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$). Again everything can be dualized. One considers the projective coordinate ring and its set of homogeneous ideals (ideals which are generated by homogeneous elements). In the case of $\mathbb{P}^n_{\mathbb{K}}$ the homogeneous coordinate ring is $\mathbb{K}[Y_0, Y_1, \dots, Y_n]$. Again it is possible to introduce the Zariski topology on the set of homogeneous prime ideals. It is even possible to introduce the notion of a projective scheme Proj, which is again a topological space together with a sheaf of rings, see [EH].

In the same way as $\mathbb{P}^n_{\mathbb{K}}$ can be covered by $(n+1)$ affine spaces $\mathbb{K}^n$ it is possible to cover every projective scheme by finitely many affine schemes. This covering is even such that the projective scheme is locally isomorphic to these affine scheme. Hence, it is a scheme. The projective scheme $\mathrm{Proj}(\mathbb{K}[Y_0, Y_1, \dots, Y_n])$ is locally isomorphic to $\mathrm{Spec}(\mathbb{K}[X_1, X_2, \dots, X_n])$. For example, the open set of elements α with $Y_0(\alpha) \neq 0$ is in 1-1 correspondence to it via the assignment $X_i \mapsto \dfrac{Y_i}{Y_0}$.

As already said, the projective schemes are schemes and you might ask why should one pay special attention to them. Projective schemes are quite useful. They are schemes with rather strong additional properties. For example, in the classical case (e.g. nonsingular varieties over $\mathbb{C}$) projective varieties are compact in the classical complex topology. This yields all the interesting results like, there are no non-constant global analytic or harmonic functions, the theorem of Riemann-Roch is valid, the integration is well-defined, and so on. Indeed, similar results we get for projective schemes. Here it is the feature "properness" which generalizes compactness.

2. The scheme of integers. The affine scheme $\mathrm{Spec}(\mathbb{Z}) = (\mathrm{Spec}(\mathbb{Z}), \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})})$ we discussed already in the last lecture. The topological space consist of the element $[\{0\}]$ and the elements $[(p)]$ where p takes every prime number. The residue fields are $\mathbb{Q}$, resp. the finite fields $\mathbb{F}_p$. What are the closed sets. By definition, these are exactly the sets $V(S)$ such that there is a $S \subseteq \mathbb{Z}$ with

$$V(S) \;:=\; \{\, [(p)] \in \mathrm{Spec}(\mathbb{Z}) \mid (p) \supseteq S\} \;=\; V((S)) \quad = \quad V((gcd(S)))\ .$$

For the last identification recall that the ideal (S) has to be generated by one element n because $\mathbb{Z}$ is a principle ideal ring. Now every element in S has to be a multiple of this

n. We have to take the biggest such n which fulfills this condition, hence $n = gcd(S)$. If $n = 0$ then $V(n) = V(0) = \mathrm{Spec}(\mathbb{Z})$, if $n = 1$ then $V(n) = V(1) = \emptyset$, otherwise $V(n)$ consists of the finitely many primes, resp. their ideals, dividing n. Altogether we get that the closed sets are beside the whole space and the empty set just sets of finitely many points. As already said at some other place of these lectures $\mathbb{Z}$ resembles very much $\mathbb{K}[X]$. By the way, we see that the topologial closure $\overline{[\{0\}]} = \mathrm{Spec}(\mathbb{Z})$ is the whole space. For this reason $[\{0\}]$ is called the generic point of $\mathrm{Spec}(\mathbb{Z})$.

All these has important consequences. We have two principles which can be very useful:

(1.) Let some property be defined over $\mathbb{Z}$ and assume it is a closed property. Assume further that the property is true for infinitely many primes (e.g. the property is true if we consider the problem in characteristic p for infinitely many p) then it has to be true for the whole $\mathrm{Spec}(\mathbb{Z})$. Especially, it has to be true for all primes and for the generic point, i.e. in characteristic zero.

(2.) Now assume that the property is an open property. If it is true for at least one point, then it is true for all points except for possibly finitely many points. In particular, it has to be true for the generic point (characteristic zero) because every non-empty open set has to contain the generic point.

3. A family of curve. This example illustrates the second principle above. To allow you to make further studies by yourself on the example I take the example from [EH]. You are encouraged to develop your own examples. Consider the conic $X^2 - Y^2 = 5$. It defines a curve in the real (or complex) plane. In fact, it is already defined over the integers which says nothing more than that there is a defining equation for the curve with integer coefficients. Hence, it makes perfect sense to ask for points $(\alpha, \beta) \in \mathbb{Z}^2$ which solve the equation. We already saw that it is advantagous to consider the coordinate ring. The coordinate ring and everything else make sense also if there would be no integer solution at all. Here we have:

$$\mathbb{Z} \;\to\; R = \mathbb{Z}[X,Y]/(X^2 - Y^2 - 5), \qquad \mathrm{Spec}(R) \;\to\; \mathrm{Spec}(\mathbb{Z}) \;.$$

We obtain an affine scheme over $\mathbb{Z}$. Now $\mathrm{Spec}(\mathbb{Z})$ is a one-dimensional base, the fibres are one-dimensional curves, and $\mathrm{Spec}(R)$ is two-dimensional. It is an arithmetic surface. We want to study the fibres in more detail. Let $Y \to X$ be a homomorphism of schemes and p a point on the base scheme X. The topological fibre over p is just the usual pre-image of the point p. But here we have to give the fibre the structure of a scheme. The general construction is as follows. Represent the point p by its residue field $k(p)$ and a homomorphism of schemes $\mathrm{Spec}(k(p)) \to X$. Take the "fibre product of schemes" of the scheme Y with $\mathrm{Spec}(k(p))$ over X. Instead of giving the general definition let me just write this down in our affine situation:

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longleftarrow & \mathrm{Spec}(R \bigotimes_{\mathbb{Z}} k(p)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(\mathbb{Z}) & \longleftarrow & \mathrm{Spec}(k(p)) \end{array} \qquad \begin{array}{ccc} R & \longrightarrow & R \bigotimes_{\mathbb{Z}} k(p) \\ \big\uparrow & & \big\uparrow \\ \mathbb{Z} & \longrightarrow & k(p) \end{array}$$

Both diagrams are commutative diagrams and are dual to each other.

Here we obtain for the generic point $[0]$ the residue field $k(0) = \mathbb{Q}$ and as fibre the Spec of

$$R \bigotimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[X, Y]/(X^2 - Y^2 - 5) \ .$$

For the closed points $[p]$ we get $k(p) = \mathbb{F}_p$ and as fibre the Spec of

$$R \bigotimes_{\mathbb{Z}} \mathbb{F}_p = \mathbb{F}_p[X, Y]/(X^2 - Y^2 - 5) \ .$$

In the fibres over the primes we just do calculation modulo p. A point lying on a curve in the plane is a singular point of the curve if both partial derivatives of the defining equation vanish at this point. Zero conditions for functions are always closed conditions. Hence non-singularity is an open condition on the individual curve. In fact, it is even an open condition with respect to the variation of the point on the base scheme. The curve $X^2 - Y^2 - 5 = 0$ is a non-singular curve over $\mathbb{Q}$. The openness principle applied to the base scheme says that there are only finitely many primes for which the fibre will become singular. Here it is quite easy to calculate these primes. Let $f(X, Y) = X^2 - Y^2 - 5$ be the defining equation. Then $\frac{\partial f}{\partial X} = 2X$ and $\frac{\partial f}{\partial Y} = 2Y$. For $p = 2$ both partial derivatives vanish at every point on the curve (the fibre). Hence every point of the fibre is a singular point. This says that the fibre over the point $[(2)]$ is a multiple fibre. In this case we see immediately $(X^2 - Y^2 - 5) \equiv (X + Y + 1)^2 \mod 2$. This special fibre is $\mathrm{Spec}(\mathbb{F}_2[X, Y]/((X + Y + 1)^2)$ which is a non-reduced scheme. For $p \neq 2$ the only candidate for a singular point is $(0, 0)$. But this candidate lies on the curve if and only if $5 \equiv 0 \mod p$ hence only for $p = 5$. In this case we get one singularity. Here we calculate that $(X^2 - Y^2 - 5) = (X + Y)(X - Y) \mod 5$. Altogether we obtain that nearly every fibre is a non-singular conic. Only the fibre over $[(2)]$ is a double line and the fibre over $[(5)]$ is a union of two lines which meet at one point.

4. Other objects. In lecture 5 we already said that moduli problems (degenerations etc.) can be conveniently be described as functors. It is not always possible to find a scheme representing a certain moduli functor. To obtain a representing geometric object it is sometimes necessary to enlarge the category of schemes by introducing more general objects like algebraic spaces and algebraic stacks. It is quite impossible even to give the basics of their definitions. Here let me only say that in a first step it is necessary to

introduce a finer topology on the schemes, the *etale* topology. With respect to the etale topology one has more open sets. Schemes are "glued" together from affine schemes using algebraic morphisms. Algebraic spaces are objects where the "glueing maps" are more general maps (etale maps). Algebraic stacks are even more general than algebraic spaces. The typical situation where they occur is in connection with moduli functors. Here one has a scheme which represents a set of certain objects. If one wants to have only one copy for each isomorphy class of the objects one usually has to divide out a group action. But not every orbit space of a scheme by a group action can be made to a scheme again. Hence we indeed get new objects. This new objects are the algebraic stacks.
Let me here only give a few references. More information on algebraic spaces you can find in the book of Artin [Ar-1] or Knutson [Kn]. For stacks the appendix of [Vi] gives a very short introduction and some examples.

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Martin Schlichenmaier, Department of Mathematics and Computer Science, University of Mannheim D-68131 Mannheim, Germany

E-mail address: schlichenmaier@math.uni-mannheim.de